Some Theoretical Considerations for a Dynamic Equation of Exchange

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The author introduces a dynamic version of the equation of exchange that is used as a basis for the derivation and discussion of (i) the relationships between force, work, and the velocity of money; (ii) the velocity of money derived from Fisher's static equation of exchange is shown to be a special case of the velocity derived from the new dynamic equation; and (iii) the economic implications of an extreme force of money on price inflation, output transactions, and the structure of the economy. Unlike previous models, this dynamic approach allows all variables to fluctuate over time, providing a more realistic framework for analyzing monetary policy impacts and economic fluctuations.

Keywords: equation of exchange, quantity theory of money, velocity of money, force of money, work of money, inflation, price bubble, output stagnancy

INTRODUCTION

Since its inception, Fisher's equation of exchange (Fisher, 1911) (hereafter referred to as the Fisherian equation) has led to many controversies in economics and politics, especially when it is considered the basis for the quantity theory of money and many decisions on monetary policies targeting inflation (Laidler, 1991).

In my view, controversies arise owing to insufficient attention to the important role of the force and the work of money in many analyses based on the Fisherian equation. The quantity theory of money has flourished from the Fisherian equation, overemphasizing the velocity of money and making extensive use of its simple derivation, together with its mathematical tautology of an identity. Unfortunately, numerous efforts to prove the Fisherian equation and its money velocity are founded on two disputable premises. First, the Fisherian equation unrealistically assumes the constancy or stability of other variables in its calculation of money velocity; and second, the Fisherian equation gives a money velocity that is too simple and is a special case of the money velocity derived from the dynamic equation of exchange (EOE), as shown in the proof given in this paper.

In this paper, a new dynamic version of EOE with all variables being allowed to be dynamic, that is, to vary with time, is proposed. The dynamic features of the new EOE enable the introduction of the force and the work of money for analysis. This inclusion, in turn, enables the derivation of a more properly specified money velocity. Some implications of the established generality of the new money velocity are also provided for future research considerations.

THE CLASSICAL EQUATION OF EXCHANGE

The Fisherian equation of exchange in transaction form is given as

$$MV = PT \quad (1)$$

where *M* is the money supply, *P* is the weighted average price of all transactions involving money, *T* is the number of transactions or volume of trade, and *V* is a variable derived from the other three variables and calculated as *V* = *PT/M.* Owing to the difficulty of collecting data for *T*, economists often use the following income form instead of (1):

$$MV = PQ \quad (2)$$

where Q is the real gross domestic product (GDP) and *P* is the GDP price deflator.

THE PROPOSED NEW DYNAMIC EQUATION OF EXCHANGE

The new dynamic equation of exchange is proposed as

$$m_t v_t = P_t Q_t \quad (3)$$

The subscript *t* indicates the value of a variable at time t hereafter.

Unlike the Fisherian equation, none of the variables in (3) are constrained by the unrealistic assumption of constancy or stability. The variables of the new dynamic equation are defined as follows:

New velocity of money v

Let $\emptyset$ be the maximum possible money velocity $\emptyset$ at which the exchange of all Q_t is performed in a chosen unit of time (such as month, quarter, year, or five years). The velocity of money v_t is defined as

$$v_t = \emptyset \sin\left(\frac{\gamma}{\emptyset} t\right) \quad (4)$$

where *t* is time, $\gamma \ll \emptyset$, and γ is a constant acceleration factor that is insignificant compared with $\emptyset$.

As specified in (4), v_t cannot exceed $\emptyset$. Additionally, at any time, a change in money velocity v_t is always counteracted by its time derivative $\frac{dv_t}{dt} = \gamma \cos\left(\frac{\gamma}{\emptyset} t\right)$. This counteraction in the movement of money is caused by changes in people's money-holding and spending habits after changes in money velocity. Money velocity can change after variations in the amount of money supply, institutional and regulatory factors, as well as opportunity costs and prices.

The effective amount of money in circulation m_t

The effective amount of money in circulation is given by m_t. This variable is related to the money supply M_t through

$$m_t = \frac{M_t}{\xi_t \cos\left(\frac{\gamma}{\emptyset} t\right)} \quad (5)$$

where M_t is defined as in (2). M_t is exogenously determined by the Central Bank; ξ_t is a variable that depends on the opportunity cost of holding money. Cos $\left(\frac{\gamma}{\emptyset} t\right)$ in the denominator reflects the effect of people's change in their attitude toward spending following a change in M followed by its effect on v_t and can be said to be more influenced by people's precautionary attitude. We can assume that M_t and ξ_t being fixed by the central bank and financial regulations.

For simplicity, we denote $x = \left(\frac{\gamma}{\emptyset} t\right)$. Then, (4) and (5) become $v_t = \emptyset \sin(x)$, and $m_t = \frac{M_t}{\xi_t \cos(x)}$ in later analysis and of v_t and m_t.

Price P_t and output Q_t

The variables P_t and Q_t are, respectively, the current price of goods and services included in the gross national product (GDP) and the real GDP, which have the same definitions as in (2). The product $P_t Q_t$ is thus the nominal GDP (GDP at current price) in the national accounts.

THE WORK OF MONEY W_t AND ITS FORCE F_t

All variables M_t, V_t, P_t and Q_t are interdependent, as shown in (3), like Equation (2). However, the main difference between Equations (2) and (3) is that the variables in (3) are not restricted by the unrealistic assumption of constancy or even stability. A variable in (3) is allowed to vary in response to changes in one or more of the other three variables. For example, a change in M_t after the decision of the Central Bank influences m_t. Such an influence will generate changes in the force of money, so v_t changes as a result. As m_t and v_t change, P_t and Q_t change depending on how efficiently money works as a means for the transaction of goods and services. The separate effects of money work on P_t and Q_t depend on the price elasticity of output in the economy. Here, we notice that the money force and work are important in the determination of money velocity and, consequently, output price and quantity. For this reason, we turn now to the work and the force of money.

The work of money $\overline{W}$ in the classical static Fisherian equation of exchange

We first review how the work of money $\overline{W}$ is conceptualized in the classical static equation of exchange.

For the whole period between time t_0 and t, the work of money $\overline{W}$ is defined as the value of money changing hand in the transactions of Q. Therefore, $\overline{W}$ is expressed as

$$\overline{W} = MV = PQ \tag{6}$$

We notice that $V_t = V$ in (6) is the average money velocity. This velocity is assumed to remain constant between t_o and t, and in this static context, V is calculated as the ratio $V = \frac{PQ}{M}$. This ratio was assumed to be constant (or at least stable) throughout the study period, which means that the other three variables P, Q, and M are constrained by the assumption of constancy or by the relationship $\log_e V = \log_e P + \log_e Q - \log_e M$. As such, money velocity and other variables in the Fisherian static equation are under strict assumptions, and more importantly, the assumption of constant or stable money velocity means that the force of money is zero or negligible. Thus, the money force is not considered in the further development of the quantity theory of money. To fill this gap, the force of money is introduced into the dynamic equation of exchange in the next subsection.

Force of money F_t in the dynamic equation of exchange

In a dynamic context, the force of money is defined as the amount of money that changes hands in one unit of time. From (3) and (4), the money force can be derived as

$$Ft{=}d(mtvt)dt = dmtdtvt{+}dvtdtmt{=}dPtdtQt{+}dQtdtPtF_t = \frac{d(m_t v_t)}{dt} = \frac{dm_t}{dt} v_t + \frac{dv_t}{dt} m_t = \frac{dP_t}{dt} Q_t + \frac{dQ_t}{dt} P_t \tag{7}$$

$$= \emptyset \frac{dm_t}{dt} \sin(x) + m_t \emptyset \frac{dx}{dt} \cos(x) \quad (8)$$

From (5), we can derive

$$\frac{dm_t}{dt} = \frac{1}{\xi_t} \frac{M_t \sin(x)}{\cos^2(x)} \frac{dx}{dt} \quad (9)$$

Note that if $\emptyset$ is so large that $(\gamma/\emptyset)t$ is small, then in approximations $sin(x) \approx x$ and $cos(x) \approx 1$, the force F is approximately equal to

$$F_t \approx \frac{1}{\xi_t} \left(\frac{M_t \sin(x)}{\cos^2(x)} \frac{dx}{dt} \emptyset \sin(x) + M_t \emptyset \frac{dx}{dt} \right) \quad (10)$$

$$= \frac{1}{\xi_t} M_t \emptyset \left(1 + \frac{\sin^2(x)}{\cos^2(x)} \right) \frac{dx}{dt} \quad (11)$$

$$\Rightarrow \quad F_t = \frac{1}{\xi_t} M_t \emptyset (1 + \tan^2(x)) \frac{dx}{dt} \quad (12)$$

In this context, the force of money refers to the rate at which money is actively used in transactions, reflecting changes in economic behavior over time. In the next subsection, we discuss the work on money.

The work of money W_t in the dynamic equation of exchange

The work done by the force of money F_t during time dt is defined as the product of F_t and the amount of money changing hands in dt. So,

$$dW_t = F_t v_t dt \quad (13)$$

Combining (11) and (13) and using $v_t = \emptyset \sin(x)$, we have

$$dW_t = \frac{1}{\xi_t} M_t \emptyset \left(1 + \frac{\sin^2(x)}{\cos^2(x)} \right) \frac{dx}{dt} v_t dt \quad (14)$$

$$= \frac{1}{\xi_t} M_t \emptyset \left(1 + \frac{\sin^2(x)}{\cos^2(x)} \right) \emptyset \sin(x) dx \quad (15)$$

We introduce the variables $u_t = cos(x) \Rightarrow du_t = -sin(x)dx$ and $\sin^2(x) = 1 - u_t^2$ to ease mathematical manipulation. By substituting u in (15),

$$dW_t = -\frac{1}{\xi_t} M_t \emptyset^2 \left(1 + \frac{1-u_t^2}{u_t^2} \right) du_t \quad (16)$$

$$= -\frac{1}{\xi_t} \frac{M_t \emptyset^2}{u_t^2} du_t \quad (17)$$

The work done over the period t_o to t is

$$\int_{t_o}^{t} \frac{dW_t}{dt} dt = \int_{t_o}^{t} -\frac{1}{\xi_t} \frac{M_t \emptyset^2}{u_t^2} du_t \quad (18)$$

This gives

$$W_t = \int_{\cos\left(\frac{\gamma}{\emptyset}t_o\right)}^{\cos\left(\frac{\gamma}{\emptyset}t\right)} -\frac{1}{\xi_t}\frac{M_t\emptyset^2}{u_t^2}du_t \tag{19}$$

We set the initial condition $t_o = 0$; hence, $x_o = \frac{\gamma}{\emptyset}t_o = 0$, and recall that $\cos\left(\frac{\gamma}{\emptyset}t\right) = \cos(x)$; then, the work done is

$$W_t = \int_0^{\cos x} -\frac{1}{\xi_t}\frac{M_t\emptyset^2}{u_t^2}du_t = \frac{1}{\xi_t}\frac{M_t\emptyset^2}{u_t}\Big]_0^{\cos x} => W_t = \frac{1}{\xi_t}M_t\emptyset^2\left(\frac{1}{\cos(x)} - 1\right) \tag{20}$$

Since $\gamma \ll \emptyset$, we have

$$\frac{v_t}{\emptyset} = \sin(x) = \sin\left(\frac{\gamma}{\emptyset}t\right) \approx \frac{\gamma}{\emptyset}t = x \tag{21}$$

$$\cos(x) = \sqrt{1-\sin^2(x)} = \sqrt{1-\frac{v_t^2}{\emptyset^2}} \approx 1 - \frac{1}{2}\frac{v_t^2}{\emptyset^2} \tag{22}$$

Substituting (22) into (20) and using $1/(1-r)$ as approximately equal to $(1+r)$ when r is small, we obtain

$$W_t = \frac{1}{\xi_t}M_t\emptyset^2\left(\frac{1}{\sqrt{1-\frac{v_t^2}{\emptyset^2}}} - 1\right) \tag{23}$$

$$\approx \frac{1}{\xi_t}M_t\emptyset^2\left(\frac{1}{1-\frac{1}{2}\frac{v_t^2}{\emptyset^2}} - 1\right) \tag{24}$$

$$\approx \frac{1}{\xi_t}M_t\emptyset^2\left(1 + \frac{1}{2}\frac{v_t^2}{\emptyset^2} - 1\right) \tag{25}$$

$$W_t = \frac{1}{\xi_t}\left(\frac{1}{2}M_t v_t^2\right) \tag{26}$$

Comparing W_t in (26) with $\overline{W}$ in (6), one would realize that $\overline{W}$ has undervalued the work of money as a means of exchange in the economy. This result also demonstrates that increases in money velocity have a disproportionately large effect on economic activity, emphasizing the importance of monitoring rapid changes in money circulation.

Force F_t and money velocity v_t

Equation (26) shows that money with its force has performed work W_t, which depends on the money supply M_t at a velocity v_t raised to the power of two. This points to the need to specify the relationship between force F_t of money and its velocity v_t.

Recall Equation (12) and define G_t as the integral of F_t over the period from time t_o to t. Then, integrating (12), we obtain

$$G_t = \int_{t_o}^{t} F_t dt \approx \frac{1}{\xi_t}M_t\emptyset \int_{t_o}^{t}\left(1+\tan^2(x)\right)\frac{dx}{dt}dt$$

$$= \frac{1}{\xi_t} M_t \emptyset \tan(x) \Big]_{t_o}^{t}$$

Again, we assume that $t_o = 0$; therefore, $, x_o = 0$, and $\tan(x_o) = 0$. Additionally, recall that $F_t = \frac{dm_t v_t}{dt}$; thus, $G_t = \frac{1}{\xi_t} M_t \emptyset \tan(x) = m_t v_t$.

Since $\sin(x) = \frac{\tan(x)}{\sqrt{1+\tan^2(x)}}$, and

$$\tan x = \frac{\xi_t G_t}{M_t \emptyset} \tag{27}$$

Therefore,

$$\sin(x) = \frac{\xi_t G_t}{\sqrt{M_t^2 \emptyset^2 + \xi_t^2 G_t^2}} \tag{28}$$

Because $v_t = \emptyset \sin(x)$, we arrive at an equation that relates v_t, M_t, t and G_t:

$$v_t = \frac{\xi_t \emptyset G_t}{\sqrt{M_t^2 \emptyset^2 + \xi_t^2 G_t^2}} \tag{29}$$

To ease the interpretation of this relationship, we define the following variable:

$$k_t = \frac{1}{t} \frac{\xi_t G_t}{M_t \emptyset} \tag{30}$$

thus

$$v_t = \frac{\emptyset}{\sqrt{1 + \frac{1}{k_t^2 t^2}}} \tag{31}$$

Equation (31) shows that if F_t strengthens, then G_t and k_t increase; thus, v_t increases, given that other variables remain unchanged, and when v_t increases, the work W_t of money increases, as shown by (26).

Furthermore, Equation (8) above and Equation (33) below show that $G_t = P_t Q_t$; thus, $k_t = \xi_t P_t Q_t / (M_t \emptyset t)$; hence, from Equation (29),

$$v_t = \frac{\emptyset}{\sqrt{1 + \frac{M_t^2 \emptyset^2}{\xi_t^2 P_t^2 Q_t^2}}} \tag{32}$$

Equation (32) shows that the stability of the money velocity v_t can be achieved if the second term under the square root in the denominator is sufficiently large in addition to the maintenance of the stability of $P_t Q_t$, and M_t. However, this is rarely the case. In other words, v_t varies between *0* and $\emptyset$, so the assumption of constant v_t is theoretically unsustainable, as shown in the dynamic equation of exchange, because it implies that $P_t Q_t$ is constant and that the money force $F_t = 0$. In short, one can say that the money velocity of the Fisherian equation is a special case of the velocity given in Equation (32).

Force F_t and price P_t and quantity Q_t

At time t, the force F_t derived from Equation (8) is

$$F_t = \frac{dP_t}{dt} Q_t + \frac{dQ_t}{dt} P_t \quad (33)$$

We can derive the changes in price level P_t and the volume of goods and services Q_t at time t as

$$\frac{dP_t}{dt} = \frac{F_t}{Q_t(1+\varepsilon)} \quad (34)$$

$$\frac{dQ_t}{dt} = \frac{F_t}{P_t\left(1+\left(\frac{1}{\varepsilon}\right)\right)} \quad (35)$$

where $\varepsilon = (dQ_t/Q_t)/(dP_t/P_t)$ is the price elasticity of volume Q_t of the goods and services.

Equation (33) shows that relationships between the force F_t, price P_t, and output Q_t can be established when the price elasticity of output Q_t is known. The price level P_t does not depend solely on v_t but also on the force F_t of money.

IMPLICATIONS OF THE RESULTS

More attention has been given to the work of money. If money is circulated too fast in an economy or just in a large sector of the economy, without any control, then one would expect a boom or bust depending on the force direction. If money velocity is too fast for output to catch up, any increase in velocity would increase the work of money by a power of two, as shown in Equation (26). In such a case, one would expect to see the quick formation of price bubbles, which, if not controlled, would burst and lead to a possible economic crisis. In the reverse direction, when money loses force, one would expect a slowdown in money velocity, and the economy would approach stagnancy if the situation were left uncontrolled.

Equation (26) also helps us understand that, in a structurally uneven economy, economic growth depends on one or only a small number of industrial sectors where money circulates much faster than the rest. Money gravitates quickly toward the money sector. Problems arise when output growth cannot keep up with fast money. Spotty price bubbles begin to form. Widespread economic problems can occur in a chain reaction fashion throughout the economy if bubbles burst.

CONCLUSION

The statement in Laidler (1991) mentions themes of the quantity theory of money, which are subjects of much controversy, including ". . . the definition of money, the relationship between correlation and causation, and the transmission mechanism. . .". In this paper, we have replaced the definition of money supply M_t with m_t, the effective amount of money in circulation that covers the rapid expansion of people's ability to take advantage of available banking credit facilities (Gardiner, 2006, Chapter 14) in their purchase of goods and services in case of need. We have also established the theoretical importance of the inclusion of money force in the dynamic equation of exchanges, its relationships with the work of money, money velocity, and its role in price and output determination.

The equations for the force of money, money work, and money velocity are related and show that a change in the exogenously determined money supply influences the effective amount of money in circulation. This produces changes in the money force, leading to changes in money speed in a dynamic economy. A change in money velocity will in turn produce a change in the value of goods and services that is equal to the change in the amount of money changing hands. With this knowledge of the transmission mechanism, policy decision-makers can search for a desirable combination of price inflation and real GDP growth, given the available estimates of the price elasticity of output.

Another application of the dynamic model is in studies of the structure of an economy. Considering an economy with large and small sectors, there are times when money starts circulating quickly in just one or just a few sectors where disruptive events such as technological changes occur. Fast money in those sectors means that their money is working hard with force much stronger than that in others. As the work of money grows, the force of money in those sectors quickly becomes increasingly powerful. They attract more money from other sectors. However, the entire economy would be in jeopardy if spotty price bubbles were formed and burst without proper controls.

ENDNOTE

1. The author, retired from the Australian Bureau of Agricultural Resource Economics, thanks Dr Nam Ho-Nguyen (Business Analytics School, University of Sydney) for highlighting the link between equations (29) and (32). The views expressed are solely those of the author.

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